## ANALYSIS QUALIFYING EXAM

JUNE 2013

### Part 1: Real analysis

Answer all 4 questions. In your proofs, you may use any major theorem, except the fact you are trying to prove (or a variant of it). State clearly what theorems you use. Good luck.

**Exercise 1.** (25 points)

(1) Prove that not every subset of $\mathbb{R}$ is Lebesgue measurable.
(2) Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of Lebesgue measurable functions. Prove that the set $E = \{x \mid \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is Lebesgue measurable.

**Exercise 2.** (25 points)
Given $f \in L^1(\mathbb{R})$ let

$$Hf(x) = \sup_{r > 0} \frac{1}{2r} \int_{x-r}^{x+r} |f(x)| dx,$$

denote the Hardy Littlewood maximal function. Show that $Hf$ is measurable and that

$$m(\{x \mid Hf(x) > \alpha\}) \leq \frac{3}{\alpha} \|f\|_1 .$$

**Exercise 3.** (25 points)
Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f \in L^\infty(X) \cap L^1(X)$. Show that $f \in L^p(X)$ for all $p > 1$ and that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

**Exercise 4.** (25 points)
Let $X$ be a Banach space and $L(X, X)$ the space of bounded linear operators.

(1) Show that the space $L(X, X)$ with the induced operator norm is also a

## Part 2: Complex analysis

In all of the following you may freely use the Cauchy integral formula and Cauchy estimates and the fact that holomorphic functions are analytic Each problem is worth 2 points

1 Let $f$ $C \to C$ be an entire holomorphic function such that $|f(z)| \leq \log(|z|)$ whenever $|z|$ is sufficiently large Show that $f$ is constant

2 Let $f$ $\{z \in C \mid |z| < 1\} \to C$ be a holomorphic function on the unit disk Suppose that $f(a_n) =$ for some nonzero sequence $a_n$ converging to Show that $f$ is identically zero Show that this need not be true if $a_n$ converges to 1

3 Give an example of a nonzero harmonic function $f$ $C \to R$ and a nonzero sequence $a_n$ converging to zero such that $f(a_n) =$ for all $n$

4 Let $f(z) = z^4 - 1$ Describe

$$\int_{-\infty}^{\infty} \frac{1}{f(x \quad ir)} dx$$

as a function of $r \in R$

5 Suppose that $f(z)$ $C \to C$ is an entire holomorphic function without any zeroes Show that there exists a holomorphic $g(z)$ such that $f(z) = e^{g(z)}$ by giving an integral formula for $g(z)$