Algebra qualifying exam

5. Let $K \subset \mathbb{C}$ be the splitting field over $\mathbb{Q}$ of the cyclotomic polynomial

$$f(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 \in \mathbb{Z}[x].$$

Find the lattice of subfields of K and for each subfield $F \subseteq K$ find polynomial $g(x) \in \mathbb{Z}[x]$ such that F is the splitting field of $g(x)$ over $\mathbb{Q}$.

6. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree five with exactly three real roots, and let K be the splitting field of f. Prove that $\mathrm{Gal}(K/\mathbb{Q}) \simeq S_5$.

7. Let k be a field, and let $R = k[x, y]/(y^2 - x^3 - x^2)$.

a) Prove that R is an integral domain.

b) Compute the integral closure of R in its quotient field.
[Hint: Let $t = \bar{y}/\bar{x}$, where $\bar{x}$ and $\bar{y}$ are the images of x and y in R.]

8. Let p be a prime and let G be the group of upper triangular matrices over the field $\mathbb{F}_p$ of p elements:

$$G = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{F}_p \right\}.$$

Let Z be the center of G and let $\rho : G \to \mathrm{GL}(V)$ be an irreducible complex representation of G. Prove the following.

a) If ρ is trivial on Z then $\dim V = 1$.

b) If ρ is nontrivial on Z then $\dim V = p$.
[Hint: Consider the subgroup of matrices in G having $y = 0$.]