A Lot of Ambiguity

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Abstract

We consider a risk averse decision maker who dislikes ambiguity as

cost, while no treatment is preferred to L. Can it be the case that eventually L^n becomes desirable? We show that this is indeed the case. Under some conditions, n repetitions of the ambiguous treatment are eventually preferred to no treatment (Theorem 2).

Should society encourage, maybe even enforce, the use of the ambiguous treatment? Patients may be willing to pay the extra price for the unambiguous treatment if it exists, or to bear the cost of no treatment if an alternative treatment does not exist. But if society adopts the point of view of social planers and care takers (even if they do not have any better information), Nau [21], Chew and Sagi [3], and Ergin and Gul [7]. For $E = \{s_{i_1}, \ldots, s_{i_\ell}\} \in$, let $P(E) = \frac{\ell}{\ }$.

Assume now the existence of a sequence of such urns. Let $S_i = S$ be the set of states in urn i with the corresponding algebra $_i =$. The information regarding each of these urns is the same. Moreover, the outcome, or even the mere existence of any urn doesn't change the decision maker's information regarding any other urn. Let S

$^{n}(?) = 0,$

$\prod_{i=1}^{n}$ Note that this is a product capacity. For all $E = E^1 \; ::: \; E^n$, ${}^n(E) = \prod_{i=1}^{n} {}^1(E^i) = 0$, unless for all i, $E^i = fG; Rg$, in which case ${}^n(E) = \prod_{i=1}^{n} {}^1(E^i) = 1$.

Following the discussion in the introduction, consider a given ambiguous act L with the anchor lottery X. Suppose that the expected value of X is zero and let X dominate a lottery Y by rst order stochastic dominance (FOSD). Theorem 1 shows that as $n \; ! \; 1$, the decision maker will prefer playing L for n times (that is, L^n) rather than playing Y for n times.

Theorem 1 Suppose that the CEU preferences satisfy ambiguity aversion, risk aversion, and boundedness. Let L be an ambiguous act with an anchor lottery X such that $\mathrm{E}(X) = 0$. Then for every Y dominated by X by strict FOSD there exists n such that for all $n > n$, $L^n \quad Y^n$.

Remark: The proof of Theorem 1 covers also the case $\mathrm{E}(X) < 0$, except for the case where $\lim_{x \,! \, 1} u^0(x) = 1$ but $\lim_{x \,! \, 1} \frac{u^{00}(x)}{u^0(x)} = 0$.

Consider now a di erent case, where $\mathrm{E}(X) > 0$. This of course doesn't mean that the decision maker accepts X, or even that if he accepts it once bu00]TJ/F32 11.9552 Tf10.85course he would accept it n act

Proposition 1 shows that under these conditions, from a certain point on the ambiguous acts L^n

Hence $\tilde{P}^n$ is in the core of ^{n} and clearly $\tilde{P}^n$ and P^n do not converge to the same limit.

Our results do not always hold without the boundedness assumption. See example 2 in the appendix. The boundedness of u from above is required for Proposition 1. See example 3 in the appendix.

4 The Smooth Model

Klibano , Marinacci, and Mukerji [16] suggested the following smooth case

As before, let X^n and L^n be n-repetitions of X and L. The value of X^n is $\mathrm{EU}^u(X^n)$. Consider L^n. A typical sequence in L^n is a list of n lotteries, each taken from the set $\{X_{p^1}, \ldots, X_{p^\ell}\}$, where X_{p^i} appears j_i times, $i = 1, \ldots, \ell$, and $\sum_i j_i = n$. The probability of such a sequence is the product of the corresponding π^i probabilities, that is, $\prod_i (\pi^i)^{j_i}$. There are $(\ell)^n$ (ℓ to the power of n) such possible sequences, denote them $\{Y_j^n\}_{j=1}^{(\ell)^n}$ and denote their corresponding probabilities π_j^n. We thus obtain that

$$\mathrm{SM}^{\varphi u}(L^n) = \sum_{j=1}^{(\ell)^n} \pi_j^n \, \varphi \circ u^{-1}(\mathrm{EU}^u(Y_j^n)) \qquad (2)$$

The next theorem shows that the results of Theorem 1 hold if the absolute measures or risk aversion of u and φ converge to the same limit as $x \to -\infty$. Observe that although $\frac{\varphi''(x)}{\varphi'(x)} \equiv \frac{u''(x)}{u'(x)}$ implies that φ is an affine transformation of u, the restriction $\lim_{x \to -\infty} \frac{\varphi''(x)}{\varphi'(x)} = \lim_{x \to -\infty} \frac{u''(x)}{u'(x)}$ does not imply that in the limit φ is an affine transformation of u. For example, let $u(x) = x$ and $\varphi(x) = x^3$ for $x < -1$.

Theorem 3 Suppose that the SM preferences satisfy ambiguity and risk aversion. Let L be an ambiguous act with an anchor lottery X such that $\mathrm{E}(X) = 0$. If $\lim_{x \to -\infty} \frac{\varphi''(x)}{\varphi'(x)} = \lim_{x \to -\infty} \frac{u''(x)}{u'(x)}$, then for every Y dominated by X by strict FOSD there exists n^* such that for all $n > n^*$, $L^n \succ Y^n$.

Proposition 1 analyzed conditions under which, within the CEU model, the acts L^n become strictly desirable. The next proposition offers conditions for a similar result under the SM model. For this, we restrict attention to the case where u represents constant absolute risk aversion. Observe that by risk aversion, $X \succ 0$ implies that $\mathrm{E}(X) > 0$.

Proposition 3 Suppose that the SM preferences satisfy ambiguity aversion and constant absolute risk aversion. If $\lim_{x \to -\infty} \frac{\varphi''(x)}{\varphi'(x)} = \lim_{x \to -\infty} \frac{u''(x)}{u'(x)}$, then for every ambiguous act L^n

risk aversion of utility function u is bounded from above [from below] by if for all x, $u''(x)/u'(x)$ is less than [more than] . The next result shows that if the degree of risk aversion of is bounded from below by $t > 0$, then for u with degree of risk aversion that is bounded from above by a sufficiently small s, if Y is sufficiently close to X then Y^n L^n, even if Y X.

Proposition 4 Let the SM preferences satisfy ambiguity and risk aversion such that the risk aversion of is bounded from below by $t >$

An event E is ambiguous if the decision maker may treat it di erently from its anchor probability. This means that if the decision maker is ambiguity averse, then the anchor probability $P^1(E)$ is not the minimal possible value of the range of the possible probabilities of E. To see why, note that if L is not a probabilistic act, then there must be at least two ambiguous events in its support. Therefore, there is a lottery $X_{\hat{q}}$ that is dominated by X by FOSD. By de nition, MEU(L) 6 EU($X_{\hat{q}}$) < EU(X).

Consider $L^n = x_1^n; E_1^n; : : : ; x_{k_n}; E_{k_n}^n$ and the corresponding anchor lottery $X^n = x_1^n; p_1^n; : : : ; x_{k_n}^n; p_{k_n}^n$ where $p_j^n =$

then the implications of Theorem 2, Proposition 1 (of the CEU model) and Proposition 3 (of the smooth model) do not hold.

Proposition 6 Let the MEU preferences satisfy risk aversion. For every ambiguous act L with an anchor lottery X such that $\mathrm{E}(X) > 0$, if there exists $\tilde{q} \in Q$ such that $\mathrm{E}(X_{\tilde{q}}) < 0$ then for a sufficiently large n, $0 \succ L^n$.

6 Discussion

As early as 1961 did William Fellner [8, pp. 678–9] ask: "there is the question whether, if we observe in him [the decision maker] the trait of nonadditivity, he is or is not likely gradually to lose this trait *as he gets used to the uncertainty with which he is faced*." Fellner pointed out a fundamental problem in answering this question empirically: In an experiment, decision makers may understand that the ambiguity is generated by a randomization mechanism and is therefore not ambiguous, but this is not necessarily the case with processes of nature or social life.

Our analysis shows that a lot depends on the way we choose to model ambiguity. But at least under some assumptions, some aspects of ambiguity aversion become insignificant when the decision maker is faced with many similar ambiguous situations within the CEU and the smooth models, and even in the maxmin model. The term "similar" is of course not well defined, but loosely speaking, our analysis shows that even though decision makers don't learn anything new about the world as they face repeated ambiguity, they may still learn not to fear this lack of knowledge.

The proofs of Theorems 1, 3, and 4 reveal another property of preferences as n increases to infinity. Denote by c^n and d^n the certainty equivalents of X^n and L^n. These theorems show that $\lim_{n \to \infty} \frac{d^n}{n} = \lim_{n \to \infty} \frac{c^n}{n}$. This interpretation of the theorems deals with the certainty equivalents per case. An alternative way to analyze attitudes per case is to divide the act L^n and the anchoring lottery X^n by n. The probabilistic lottery will then converge to its average. Maccheroni and Marinacci [18] proved that as

Appendix A: Proofs

Given the anchor lottery $X^n = (x_1^n, p_1^n; \ldots; x_{k_n}^n, p_{k_n}^n)$, define $g^n : [0,1] \to [0,1]$ such that for $i = 1, \ldots, k_n$,

$$g^n\left(\sum_{j=1}^{i} p_j^n\right) = 1 - \nu^n\left(\bigcup_{j=i+1}^{k_n} E_j^n\right)$$

and let g^n be piecewise linear on the segment $[0, p_{k_n}^n]$ and on the segments $[\sum_{j=1}^{i} p_j^n, \sum_{j=1}^{i+1} p_j^n]$, $i = 1, \ldots, k_n - 1$. Note that by ambiguity aversion for all E, $\nu^n(E) \leqslant P^n(E)$, hence by the piece-wise linearity of g^n, we have $g^n(p) \geqslant p$. Eq. (1) thus becomes

$$\mathrm{CEU}^n(L^n) = u(x_1^n) g^n(p_1^n) + \sum_{i=2}^{k_n} u(x_i^n)\left[g^n\left(\sum_{j=1}^{i} p_j^n\right) - g^n\left(\sum_{j=1}^{i-1} p_j^n\right)\right]$$

Denote by F_Z the distribution of lottery

hence by inequality (3), for sufficiently large n,

$$c^n - d^n \le \frac{u(c^n) - u(d^n)}{u'(c^n)} \le \frac{-Ku(c^n)}{u'(c^n)}$$

By l'Hopital's rule, since $\lim_{x\to-\infty} u(x) = -\infty$ and $\lim_{x\to-\infty} u'(x) = \infty$, $\lim_{x\to-\infty} \frac{u'(x)}{u(x)} = \lim_{x\to-\infty} -\frac{u''(x)}{u'(x)} = a > 0$. By Lemma 4, $\lim_{n\to\infty} c^n = -\infty$, hence for a sufficiently large n,

$$\frac{Ku(c^n)}{u'(c^n)} \le \frac{K+1}{a} \Rightarrow 0 \le \frac{c^n}{n} - \frac{d^n}{n} \le \frac{K+1}{an} \xrightarrow[n\to\infty]{} 0$$

It thus follows that $\lim_{n\to\infty} \frac{d^n}{n} = \lim_{n\to\infty} \frac{c^n}{n}$.

Denote this common limit $\hat{c}$. By Lemma 5, $\hat{c}$ is the certainty equivalent of X under v, where $v(x) = x$ if $a = 0$, and $v(x) = -e^{-ax}$ if $a > 0$. Consider Y dominated by X by strict FOSD, and let $\hat{b} < \hat{c}$ be the certainty equivalent of Y under v. Let b^n be the certainty equivalent of Y^n under u. By Lemma 5, $\lim_{n\to\infty} \frac{b^n}{n} = \hat{b}$, hence $\lim_{n\to\infty} \frac{b^n}{n} < \lim_{n\to\infty} \frac{d^n}{n}$. It thus follows that for sufficiently large n, $d^n > b^n$, hence $L^n \succ Y^n$.

Proof of Theorem 2: Assume wlg that $u(0) = 0$, $u'(0) = 1$, that $n_0 = 1$, and hence $c^n > 0$ for all n. Assume first that $\lim_{x\to-\infty} u'(x) = \infty$. Define $u^n(x) = u(x) - u(nx_m)$ and note that $u^n(nx_m) = 0$ and $u^n(x) < 0$, for all outcomes of X^n. These inequalities and the boundedness assumption imply that for the $\mathrm{CEU}^n_{u^n}$, the CEU^n functional with respect to u^n,

$$\begin{aligned} u^n(d^n) = \mathrm{CEU}^n_{u^n}(L^n) &= \int u^n(z)\, dg^n(F_{X^n}(z)) \\ &\geq K \int u^n(z)\, dF_{X^n}(z) \geq Ku^n(c^n) \end{aligned}$$

The inequality $u^n(c^n) \geq u^n(0)$ yields $u^n(d^n) \geq Ku^n(0)$.

Going back to u, noting that $1 - K \le 0$ and that, by concavity, $u(nx_m) \le nu(x_m)$,

$$\begin{aligned} u(d^n) &= u^n(d^n) + u(nx_m) \geq Ku^n(0) + u(nx_m) \\ &= -Ku(nx_m) + u(nx_m) = (1-K)u(nx_m) \\ &\geq n(1-K)u(x_m) \end{aligned}$$

Denote $A = (1 - K)u(x_m)$. By assumption, $A \neq 0$. Note that the concavity of u and $\lim_{x \to \infty} u'(x) = \infty$ imply $\lim_{y \to \infty} u^{-1}(y)/y = 0$. Then, $d^n > u^{-1}(nA)$ implies $\lim_{n \to \infty} \frac{d^n}{n} > \lim_{n \to \infty} \frac{u^{-1}(nA)}{nA} A = 0$.

Finally, if $\lim_{x \to \infty} u'(x) = H < \infty$

and

$$
\begin{aligned}
CEU^n\left(L\left(\frac{d^n}{n}\right)^n\right) &= \int e^{\;az}\,dg^n\left(F_{\left(X\;\frac{d^n}{n}\right)^n}(z)\right) = \\
\int e^{\;az}\,dg^n\left(F_{X^n\;d^n}(z)\right) &= \int e^{\;a(z\;d^n)}\,dg^n\left(F_{X^n}(z)\right) = \qquad (5) \\
e^{ad^n}\int e^{\;az}\,dg^n\left(F_{X^n}(z)\right) &= e^{ad^n}\,e^{\;ad^n} = 1
\end{aligned}
$$

The sequence $\left\{\frac{d^n}{n}\right\}_{n=1}^{\infty}$ is bounded (since the support of X is) and assume, by way of negation, that the sequence does not converge to c^1. Then, wlg there exists $\varepsilon > 0$ and a subsequence $\left\{\frac{d^{n_j}}{n_j}\right\}_{j=1}^{\infty}$ satisfying $\lim_{j\to\infty}\frac{d^{n_j}}{n_j} < c^1\;\varepsilon$. Without loss of generality, assume that for all j, $\frac{d^{n_j}}{n_j} < c^1\;\varepsilon$. Hence,

$$
\begin{aligned}
CEU^n\left(L\left(\frac{d^{n_j}}{n_j}\right)^{n_j}\right) &= \int e^{\;az}\,dg^n\left(F_{\left(X\;d^{n_j}/n_j\right)^{n_j}}(z)\right) \\
&> \int e^{\;az}\,dg^n\left(F\;\;{}_d\right.
\end{aligned}
$$

The ratio between this difference and 2^{-n}, the probability of $E^{n'}$, is $\frac{\sqrt{\ }}{2^n}$, which is not bounded by any K.

For Theorem 1, consider the ambiguous act $L = (-0.5; E_1; 0.5; E_2)$ with the anchor lottery $X = (-0.5; \frac{1}{2}; 0.5; \frac{1}{2})$. Let $Y = (-0.55; \frac{1}{2}; 0.45; \frac{1}{2})$. The certainty equivalent of Y^n is $-0.17n$ and that of L^n is $-0.21n$.

For the other results, consider the act $L = (-.35; E_1; 0.65; E_2)$ with the anchor lottery $X = (-0.35; \frac{1}{2}; 0.65; \frac{1}{2})$ and let $Y = (-0.02; 1)$. The certainty equivalent of X^n is $0.03n$, while that of Y^n is $-0.02n > -0.06n$, which is larger than the certainty equivalent of L^n.

Example 3 The boundedness of u from above is required for Proposition 1. Let $X = (-\frac{1}{4}; \frac{1}{2}; \frac{3}{4}; \frac{1}{2})$. Define L^n as in example 1. We get

$$\mathrm{EU}(X^{4n}) = \sum_{i=-n}^{3n} \binom{4n}{i+n} \frac{1}{2^{4n}} u(i) \tag{6}$$

$$\mathrm{CEU}^n(L^{4n}) = 2\sum_{i=-n}^{n-1} \binom{4n}{i+n} \frac{1}{2^{4n}} u(i) + \binom{4n}{2n} \frac{1}{2^{4n}} u(n) \tag{7}$$

Let $u(x) = x$ for $x > 0$. We define $u(-n)$ inductively. Let

$$v_n = \sum_{i=-n+1}^{-1} \binom{4n}{i+n} u(i) - \sum_{i=1}^{n-1} \binom{4n}{i+n} i - \binom{4n}{2n} \frac{n}{2} \tag{8}$$

$$w_n = 2u(-n+1) - u(-n+2)$$

and define u for $x < 0$ as follows. For $n = 1, \ldots,$ let $u(-n) = \min\{v_n, w_n\}$, and for $x \in (-n, -n+1)$ let $u(x) = u(-n) + (x+n)[u(-n+1) - u(-n)]$. The function u is strictly increasing and weakly concave.

Claim 1 $\lim_{n \to \infty} u(-n)/n = -\infty$.

Proof: Suppose not. Then there exists $A > 0$ such that for all n, $-u(-n)/n \leq A$, and since between $-n$ and $-n+1$ the function u is linear, it follows that for all n, , , , , , By definition, $u(-n) \le v_n$, hence it follows by eqs. (7) and (8) that for all n, $\mathrm{CEU}^n(X^{4n}) \le 0$. On the other hand, by eq. (7),

$$\begin{aligned}\mathrm{CEU}^n(X^{4n}) &= 2\sum_{i=-n}^{-1}\binom{4n}{i+n}\frac{u(i)}{2^{4n}} + 2\sum_{i=1}^{n-1}\binom{4n}{i+n}\frac{i}{2^{4n}} + \binom{4n}{2n}\frac{n}{2^{4n}} \\ &\ge \frac{(n-1)nA}{2^{4n-1}}\binom{4n}{n-1} + 1 - \frac{1}{2}\Pr(X^{4n}\le 0) \qquad (9)\end{aligned}$$

Let $\alpha_n = \frac{(n-1)nA}{2^{4n-1}}\binom{4n}{n-1}$. Clearly

$$\begin{aligned}\frac{\alpha_{n+1}}{\alpha_n} &= \frac{n(n+1)A2^{4n-1}\binom{4n+4}{n}}{(n-1)nA2^{4n+3}\binom{4n}{n-1}} \\ &= \frac{(n+1)(4n+4)(4n+3)(4n+2)(4n+1)}{16(n-1)n(3n+4)(3n+3)(3n+2)} \to \frac{4^4}{16\cdot 3^3} = \frac{16}{27}\end{aligned}$$

Hence $\lim_{n\to\infty}\alpha_n = 0$. Likewise, $\Pr(X^{4n}\le 0) \le \frac{n}{2^{4n}}\binom{4n}{n} \to 0$, hence the expression of eq. (9) converges to $\frac{1}{2}$; a contradiction.

Define $n_0 = 0$, and let $n \to$

Proof of Theorem 3: The certainty equivalents are defined by $u(c^n) = \mathrm{EU}^u(X^n)$ and $\;(d^n) = \mathrm{SM}\;^u(L^n)$.[8] By ambiguity aversion, is more concave than u, hence $\mathrm{SM}\;(L^n) \leq \mathrm{SM}\;^u(L^n) \leq \mathrm{SM}^{uu}(L^n)$. Let d^n be the certainty equivalent of L^n under SM and note that c^n is the certainty equivalent of SM^{uu} (since $\mathrm{SM}^{uu}(L^n) = \mathrm{EU}^u(X^n)$). Hence $d^n \leq d^n \leq c^n$ for all n and

$$\lim_{n \to \infty} \frac{d^n}{n} \leq \lim_{n \to \infty} \frac{d^n}{n} \leq \lim_{n \to \infty} \frac{c^n}{n}$$

Using $\mathrm{SM}\;(L^n) = \mathrm{EU}\;(X^n)$, Lemma 5 implies $\lim_{n \to \infty} \frac{\bar{d}^n}{n} = \lim_{n \to \infty} \frac{c^n}{n}$. Hence, $\lim_{n \to \infty} \frac{d^n}{n} = \lim_{n \to \infty} \frac{c^n}{n}$. The rest of the proof is similar to the last paragraph in the proof of Theorem 1.

Proof of Proposition 3

where the limit is 0 because $\mathrm{EU}^{v_t}(X) \in (-\infty; 0)$ and $\lim_{n\to\infty} \Pr(y \le X^n < 0) = 0$. As $\lim_{n\to\infty} \mathrm{EU}\ (X^n)_{x>0} = \sup_x\ (x)$, we conclude that for sufficiently large n, $\mathrm{EU}\ (X^n) > \ (0)$ and $L^n \ \ 0$.

Proof of Proposition 4: If the risk aversion of is bounded from below by t and u is concave, then for every n, $d^n_u \le d^n$, where d^n_u is the certainty equivalent of L^n under u and and d^n is the certainty equivalent of L^n under the functions $u(x) = x$ and $(x) = \ e^{\ tx}$.

Denote $z_i = \mathrm{E}(X_{p^i})$, $Z = (z_1; \ _1; \ldots; z_\ell; \ _\ell)$ and note that

$$\mathrm{E}(Z) = \sum_{i=1}^{\ell} \ _i \mathrm{E}(X_{p^i}) = \mathrm{E}\left(\sum_{i=1}^{\ell} \ _i X_{p^i} \right) = \mathrm{E}(X) = 0$$

If the decision maker is using and u, then

$$\begin{aligned} \mathrm{SM}\ ^{u}(L) &= \sum_{i=1}^{\ell} \ _i \quad u^{-1}(\mathrm{EU}^{u}(X_{p^i})) = \sum_{i=1}^{\ell} \ _i \ (\mathrm{E}(X_{p^i})) \\ &= \sum_{i=1}^{\ell} \ _i \ (z_i) = \mathrm{EU}\ (Z) \end{aligned}$$

Also, it follows from eq. (2) that

$$\mathrm{SM}\ ^{u}(L^n) = \sum_{j=1}^{n} \ ^n_j \quad u^{-1}(\mathrm{EU}^{u}(Y^n_j)) = \sum_{j=1}^{n} \ ^n_j \ [\mathrm{E}(Y^n_j)]$$

The expected value of Y^n_j is the sum of the expected values of the sequence of lotteries it represents. As there are in this sequence j_i lotteries of type X_{p^i}, $i = 1, \ldots, \ell$, the expected value of Y^n_j is $\sum_{i=1}^{\ell} j_i \mathrm{E}(X_{p^i})$. Hence

$$\begin{aligned} \sum_{j=1}^{n} \ ^n_j \ [\mathrm{E}(Y^n_j)] &= \sum_{j=1}^{n} \ ^n_j \ \left[\sum_{i=1}^{\ell} j_i \mathrm{E}(X_{p^i}) \right] \\ &= \sum_{j=1}^{n} \ ^n_j \ \left[\sum_{i=1}^{\ell} j_i z_i \right] = \mathrm{EU}\ (Z^n) \end{aligned}$$

Assume rst that is exponential of the form $(x) = \quad e^{\ tx}$. If u is linear, then the proof of the rst part of Proposition 4 implies $\frac{d^n}{n} = d^1 < 0 = \frac{c^n}{n}$. Next, consider exponential $u(x) = \quad e^{\ sx}$ where, by assumption, $s > 0$. Since $t > s$, $h(y) = \quad (\ y)^{t=s}$ is strictly concave and increasing. Then, the above equations imply $u(d^1) < u(c^1)$ and $d^1 < c^1$.

By Lemma 1, $\frac{c^n}{n} = c^1$ for all n and hence $\lim_{n\,!\,1} \frac{c^n}{n} = c^1$. Moreover, denoting $c_i = u^{\ 1}(\mathrm{EU}^u(X_{p^i}))$ and using Lemma 1, for any sequence of lotteries $Y_u^n = (X_{p^1})^{n^1}; :::; (X_{p^`})^{n^`}$, $n^i\ 2\ f0; Ng$,

$$\mathrm{EU}^u((X_{p^1})^{n^1} ::: (X_{p^`})^{n^`}) = \quad j\,\mathrm{EU}^u(X_{p^1})\,j^{n^1} \quad ::: \quad j\,\mathrm{EU}^u(X_{p^`})\,j^{n^`} =$$

$$e^{\ sc_1\ \ n^1} \quad ::: \quad e^{\ sc_`\ \ n^`} = \quad e^{\ s(n^1c_1+:::+n^`c_`)} = u(n^1c_1 + ::: + n^`c_`)$$

Therefore, denoting $C = (c_1; \ ^1; :::; c_`; \ ^`)$, SM $^u(L^n)$ can be written as EU (C^n)nX11..4[2 2 11.9552 Tf (note that $\hat{c}(s) = -\frac{1}{s}\ln(\sum p_i e^{-sx_i})$). Using l'Hopital's rule we get

$$\lim_{s\to\infty}\hat{c}(s) = \lim_{s\to\infty}\frac{\sum p_i x_i e^{-sx_i}}{\sum p_i e^{-sx_i}} = \lim_{s\to\infty}\frac{p_1 x_1 + \sum_{i>1} p_i x_i e^{-s(x_i - x_1)}}{p_1 + \sum_{i>1} p_i e^{-s(x_i - x_1)}} = x_1$$

which, noting that $c^n > nx_1$ and hence $\frac{c^n}{n} > x_1$, implies $\lim_{n\to\infty}\frac{c^n}{n} = x_1$. Similarly, for $Y = X - \varepsilon$, the certainty equivalent b^n of Y^n satisfies $\lim_{n\to\infty}\frac{b^n}{n} = x_1 - \varepsilon$. Now $d^n > nx_1$ implies $\lim_{n\to\infty}\frac{b^n}{n} = x_1 - \varepsilon < x_1 \leqslant \lim_{n\to\infty}\frac{d^n}{n}$, hence for a sufficiently large n, $L^n \succ Y^n$.

Next, consider the case $\lim_{x\to-\infty} -\frac{u''(x)}{u'(x)} = a \in (0, \infty)$. By Lemma 5 case (iii), $\lim_{n\to\infty}\frac{c^n}{n} = \hat{c}$ where $\hat{c}$ is the certainty equivalent of X under the utility $v(x) = -e^{-ax}$. Let $\hat{q} \in Q$ be a probability vector such that X strictly FOSD dominates $X_{\hat{q}}$ and let $\hat{d}$ denote the certainty equivalent of $X_{\hat{q}}$ under v. Clearly, $\hat{d} < \hat{c}$. Define $\hat{d}^n = u^{-1}(\mathrm{EU}\ X_{\hat{q}}^n$

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throughout that $\mathrm{E}(X) \neq 0$. In Lemmas 3–6 we assume
all utility functions is zero at zero and that their deriva

Lemma 1 Let $u(x) = -e^{-ax}$. Then for lotteries $X_1, \dots$
$u(\sum_{i=1}^{k}\mathrm{CE}(X_i))$, where $\mathrm{CE}(X)$ is the certainty equival
lar, if $X_i = X$ for all i, then for all n, $\frac{c^n}{n} = c^1$.

Proof: The proof follows from a property of the moment generating functions (see Bulmer [1]).

Lemma 2 There exists n_0 such that for all $n > n_0$, $\int_{z \leqslant 0} z \, dF_{X^n}(z) > \frac{x_1 \quad 2}{n^{2(\quad 1)}}$ $n \quad + n\mathrm{E}(X)$

Proof: As ^{2} be the variance of X, n ^{2} is the variance of X^n. Choose $\frac{1}{2}2$

$$\frac{\int_{x>0} x\,dF_{X^n}(x)}{\int_{x<0} u(x)\,dF_{X^n}(x)} > \frac{\int_{y(\ )}^{0} x\,dF_{X^n}(x) + \int_{x<y(\ )} x\,dF_{X^n}(x)}{\int_{y(\ )}^{0} u(x)\,dF_{X^n}(x) + \int_{x<y(\ )} u(x)\,dF_{X^n}(x)} > \frac{\int_{y(\ )}^{0} x\,dF_{X^n}(x) + \int_{x\le y(\ )} x\,dF_{X^n}(x)}{\int_{y(\ )}^{0} u(x)\,dF_{X^n}(x) + \int_{x<y(\ )} x\,dF_{X^n}(x)} \underset{n\to\infty}{\to} \frac{1}{\ }$$

This is true for every $\ > 1$, hence the claim.

Conclusion 1 If $\lim_{x\to-\infty} u'(x) = \infty$, and if for all $x < M$, $u(x) = v(x)$, then $\lim_{n\to\infty} \frac{c_u^n}{n} = \lim_{n\to\infty} \frac{c_v^n}{n}$.

Proof: For $M > 0$, the fact follows from Lemma 3. For $M < 0$, it follows by Lemma 3 and by the Central Limit Theorem (observe that $\lim_{n\to\infty} \Pr(X^n \in [M;0]) = 0$).

Lemma 4 If $\lim_{x\to-\infty} u'(x) = \infty$, then $\lim_{n\to\infty} c^n = -\infty$.

Proof: By risk aversion, $c^n \le E(X^n) = nE(X)$. Therefore, if $E(X) < 0$, we are through. If $E(X) = 0$, we show that for every integer $m < 0$, $\lim_{n\to\infty} EU(X^n) \le u(m-1)$. The value of $EU(X^n)$ equals

$$\int_{x\le 2(m-1)} u(x)\,dF_{X^n}(x) \left[1 + \frac{\int_{2(m-1)}^{0} u(x)\,dF_{X^n}(x)}{\int_{x\le 2(m-1)} u(x)\,dF_{X^n}(x)} + \frac{\int_{x>0} u(x)\,dF_{X^n}(x)}{\int_{x\le 2(m-1)} u(x)\,dF_{X^n}(x)} \right]$$

As in the proof of Lemma 3, it follows by the central limit theorem that $\lim_{n\to\infty} \int_{2(m-1)}^{0} u(x)\,dF_{X^n}(x) = 0$ and

$$\lim_{n\to\infty} \frac{\int_{x>0} u(x)\,dF_{X^n}(x)}{\int_{x\le 2(m-1)} u(x)\,dF_{X^n}(x)} = \lim_{n\to\infty} \frac{\int_{x>0} u(x)\,dF_{X^n}(x)}{\int_{x\le 0} u(x)\,dF_{X^n}(x)} = 0$$

where the last equality follows by Lemma 3. By the Central Limit Theorem, the probability of receiving a negative outcome is [1]

Hence,

$$
\begin{aligned}
\lim_{n \to \infty} \frac{c^n}{n} &= \lim_{n \to \infty} \frac{c^n}{u(c^n)} \frac{u(c^n)}{n} = \frac{1}{H} \lim_{n \to \infty} \frac{u(c^n)}{n} \\
&> \frac{1}{H} \lim_{n \to \infty} \frac{H}{n} \quad \frac{x_1 \quad ^2}{n^{2(\quad 1)}} \quad n \quad + n\mathrm{E}(X) \\
&= \lim_{n \to \infty} \quad \frac{x_1 \quad ^2}{n}
\end{aligned}
$$

monotonically increasing towards H when $x \to -\infty$) and hence $\lim_{x \to -\infty} \frac{u''(x)}{u'(x)} = 0$, contradicting $a > 0$.

For any $\varepsilon > 0$ denote $v_{\varepsilon+}(x) = -e^{-(a+\varepsilon)x}$, $v_{\varepsilon-}(x) = -e^{-(a-\varepsilon)x}$ and let $\hat{c}_{\varepsilon+}$ and $\hat{c}_{\varepsilon-}$ satisfy

$$e^{-a\hat{c}_{\varepsilon+}} = \int e^{-(a+\varepsilon)z}\, dF_X(z); \qquad e^{-a\hat{c}_{\varepsilon-}} = \int e^{-(a-\varepsilon)z}\, dF_X(z)$$

Since $v_{\varepsilon+}$ is more concave than v and v is more concave than $v_{\varepsilon-}$, we have $\hat{c}_{\varepsilon+} < \hat{c} < \hat{c}_{\varepsilon-}$. Let $\hat{c}^n_{\varepsilon+}$ and $\hat{c}^n_{\varepsilon-}$ denote the certainty equivalents of X^n under $v_{\varepsilon+}$ and $v_{\varepsilon-}$, respectively. By Lemma 1, $\lim_{n\to\infty} \frac{\hat{c}^n_{\varepsilon+}}{n} = \hat{c}_{\varepsilon+}$ and $\lim_{n\to\infty} \frac{\hat{c}^n_{\varepsilon-}}{n} = \hat{c}_{\varepsilon-}$.

As $\lim_{x\to-\infty} \frac{u''(x)}{u'(x)} = a > 0$, for every $a > \varepsilon > 0$ there is $x(\varepsilon)$ such that for all $x \le x(\varepsilon)$, $a - \varepsilon < \frac{u''(x)}{u'(x)} < a + \varepsilon$. Define the functions $u_{\varepsilon\sigma}$, $\sigma = +; -$, by

$$u_{\varepsilon\sigma}(x) = \begin{cases} u(x) & x \le x(\varepsilon) \\ \alpha v_{\varepsilon\sigma}(x) + \beta & \text{otherwise} \end{cases}$$

where $\alpha = \frac{u'(x(\varepsilon))}{v'_{\varepsilon\sigma}(x(\varepsilon))}$ and $\beta = u(x(\varepsilon)) - \alpha v_{\varepsilon\sigma}(x(\varepsilon))$ are defined as to enable continuity and differentiability of these functions.

Clearly, $u_{\varepsilon-}$ is more risk averse than $v_{\varepsilon-}$ and $u_{\varepsilon+}$ is less risk averse than $v_{\varepsilon+}$. Hence, $c^n_{u_{\varepsilon+}}$ and $c^n_{u_{\varepsilon-}}$, the certainty equivalents of X^n under $u_{\varepsilon+}$ and $u_{\varepsilon-}$, respectively, satisfy $\hat{c}^n_{\varepsilon-} > c^n_{u_{\varepsilon-}}$ and $c^n_{u_{\varepsilon+}} > \hat{c}^n_{\varepsilon+}$. Hence,

$$\hat{c}_{\varepsilon-} = \lim_{n\to\infty} \frac{\hat{c}^n_{\varepsilon-}}{}$$

$-\frac{u''(x)}{u'(x)} < s < t < -\frac{v''(x)}{v'(x)}$. Then

$$\begin{aligned} \ln(u'(0)) - \ln(u'(x)) \leq sx \quad &\text{and} \quad \ln(v'(0)) - \ln(v'(x)) \geq tx \implies \\ \ln(u'(x)) \geq \ln(u'(0)) - sx \quad &\text{and} \quad \ln(v'(x)) \leq \ln(v'(0)) - tx \implies \\ u'(x) \geq u'(0)e^{-sx} \quad &\text{and} \quad v'(x) \leq v'(0)e^{-tx} \implies \\ u(x) \geq u(0) - u'(0)e^{-sx} \quad &\text{and} \quad v(x) \leq v(0) - v'(0)e^{-tx} \implies \\ u(x) - v(x) \quad \geq \quad u(0) - v(0) &- [u'(0)e^{-sx} - v'(0)e^{-tx}] \end{aligned}$$

As $x \to -\infty$, the rhs converges to ∞, hence the claim.

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